

OBSERVABILITY AND DETECTABILITY OF LINEAR SWITCHING SYSTEMS: A STRUCTURAL APPROACH

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ABSTRACT. We define observability and detectability for linear switching systems as the possibility of reconstructing and respectively of asymptotically reconstructing the hybrid state of the system from the knowledge of the output for a suitable choice of the control input. We derive a necessary and sufficient condition for observability that can be verified computationally. A characterization of control inputs ensuring observability of switching systems is given. Moreover, we prove that checking detectability of a linear switching system is equivalent to checking asymptotic stability of a suitable switching system with guards extracted from it, thus providing interesting links to Kalman decomposition and the theory of stability of hybrid systems.

1. INTRODUCTION

Research in the area of hybrid systems addresses significant application domains with the aim of developing further understanding of the implications of the hybrid model on control algorithms and to evaluate whether using this formalism can be of substantial help in solving complex, real-life, control problems. In many application domains, hybrid controller synthesis problems are addressed by assuming full hybrid state information, although in many realistic situations state measurements are not available. Hence, to make hybrid controller synthesis relevant, the design of hybrid state observers is of fundamental importance. A step towards a procedure for the synthesis of these observers is the analysis of observability and detectability of hybrid systems. Observability has been extensively studied both in the continuous [11, 13] and in the discrete domains (see e.g. [18, 19]). In particular, Sontag in [20] defined a number of observability concepts and analyzed their relations for polynomial systems. More recently, various researchers investigated observability of hybrid systems. The definitions of observability and the criteria to assess this property varied depending on the class of systems under consideration and on the knowledge that is assumed at the output. Incremental observability was introduced in [4] for the class of piecewise affine systems. Incremental observability implies that different initial states always give different outputs independently of the applied input. A characterization of observability and the definition of a hybrid observer for the class of autonomous piecewise affine systems can be found in [6]. In [10] observability of autonomous hybrid systems was analyzed by using abstraction techniques. In [2], the notion of generic final-state determinability proposed in [20] was extended to hybrid systems and sufficient conditions were given for linear hybrid systems. The work in [22] considered autonomous switching systems and proposed a definition of observability based on the concept of indistinguishability of continuous initial states and discrete state evolutions from the outputs in free evolution. In [8, 1] observability of switching systems (with control) was investigated. Critical observability for safety critical switching systems was introduced in [7], where a set of “critical” states must be reconstructed immediately since they correspond to hazards that may yield catastrophic events.

While observability of hybrid systems was addressed in the papers cited above, a general notion of detectability has not been introduced as yet. To the best of our knowledge, the only contribution dealing with detectability can be found in [16] where detectability was defined for the class of jump linear systems as equivalent to the existence of a set of linear gains ensuring the convergence to zero of the estimation error in a stochastic setting. In this paper we address observability and detectability for the class of switching systems. General notions of

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observability and detectability are introduced for the class of linear switching systems, though our definitions apply to more general classes of hybrid systems, since they involve only dynamical properties of the executions that are generated by the hybrid system. Further, we derive a computable necessary and sufficient condition for assessing observability. We then characterize detectability using a Kalman-like approach. In particular, we show that checking detectability of a linear switching system is equivalent to checking asymptotic stability of a suitable linear switching system with guards associated with the original system. This result is clearly related to the classical detectability analysis of linear systems. It is important because it allows one to leverage a wealth of existing results on the stability of switching and hybrid systems (see e.g. [17, 5, 12] and the references therein). A preliminary version of this paper appeared before in [8]. A characterization of observability, close to the one of [8] and of the one presented in this paper, can be found in [1] for a subclass of the switching systems considered in [8]. The relation between [1], [8] and the present paper is discussed in Section 3.

The paper is organized as follows. In Section 2, we introduce linear switching systems and the notions of observability and detectability. Section 3 is devoted to finding conditions for the reconstruction of the discrete component of the hybrid state. In Section 4 we give a characterization of observability and detectability. In Section 5, an example shows the applicability and the benefits of our results. Section 7 includes technical proofs of some of the results established in Section 3. Section 6 offers some concluding remarks.

2. PRELIMINARIES AND BASIC DEFINITIONS

In this section, we introduce the notations and some basic definitions that are used in the paper.

2.1. Notation. The symbols \mathbb{N} , \mathbb{R} and \mathbb{R}^+ denote the natural, real and positive real numbers, respectively. The symbol I denotes the identity matrix of appropriate dimensions. Given a vector $x \in \mathbb{R}^n$, the symbol x' denotes the transpose of x . The symbol $\|\cdot\|_n$ denotes the Euclidean norm of a vector in the linear space \mathbb{R}^n . Given a linear subspace H of \mathbb{R}^n , the symbol $\dim(H)$ denotes its dimension and the symbol π_H denotes the projector on H , i.e. $\pi_H x$ is the Euclidean orthogonal projection of x onto H . Given a matrix $M \in \mathbb{R}^{n \times m}$, the symbols $\text{Im}(M)$ and $\ker(M)$ denote respectively the range and the null space of M ; given a set $H \subseteq \mathbb{R}^n$ the symbol $M^{-1}(H)$ denotes the inverse image of H through M , i.e. $M^{-1}(H) = \{x \in \mathbb{R}^m \mid \exists y \in H : y = Mx\}$. Given a set Ω , the symbol $\text{card}(\Omega)$ denotes the cardinality of Ω .

2.2. Switching systems. We consider the class of linear switching systems and the class of linear switching systems with guards, which generalize the class defined in [8], following the general model of hybrid automata of [15, 21]. Switching systems are relevant in many application domains such as, among many others, mechanical systems, power train control, aircraft and air traffic control, switching power converters, see e.g. [12, 7, 9] and the references therein.

The hybrid state ξ of a *GLSw*-system \mathcal{H} is composed of two components: the discrete state i belonging to the finite set $Q = \{1, 2, \dots, N\}$, called discrete state space, and the continuous state x belonging to the linear space \mathbb{R}^{n_i} , whose dimension n_i depends on $i \in Q$. The hybrid state space of \mathcal{H} is then defined by $\Xi = \bigcup_{i \in Q} \{i\} \times \mathbb{R}^{n_i}$. The control input of \mathcal{H} is a function $u \in \mathcal{U}$, where \mathcal{U} denotes the class of piecewise continuous functions $u : \mathbb{R} \rightarrow \mathbb{R}^m$. The output function of \mathcal{H} belongs to the set \mathcal{Y} of piecewise continuous functions $y : \mathbb{R} \rightarrow \mathbb{R}^l$. The evolution of the continuous state x and of the output y of \mathcal{H} is determined by the linear control systems:

$$(2.1) \quad S(i) : \begin{cases} \dot{x} = A_i x + B_i u, \\ y = C_i x, \end{cases}$$

whose dynamical matrices A_i, B_i, C_i depend on the current discrete state $i \in Q$. The evolution of the discrete state of \mathcal{H} is governed by a Finite State Machine (FSM), so that a transition from a state $i \in Q$ to a state $h \in Q$ may occur if $e = (i, h) \in E$, where $E \subseteq Q \times Q$ is the set of (admissible) transitions in the FSM, and if the continuous state x is in the set $G(e) \subseteq \mathbb{R}^{n_i}$, called guard¹ [14]. Whenever a transition $e = (i, h)$ occurs, the continuous state x is instantly reset to a new value $R(e)x$, where R is the reset function which associates

¹In this paper, the role of the guard $G(e)$ is to enable (and not to enforce) a transition.

a matrix $R(e) \in \mathbb{R}^{n_h \times n_i}$ to each $e \in E$. We assume that $R(e) \neq I$, for any in-loop transition $e = (i, i) \in E$. A *linear switching system with guards* (*GLSw-system*) \mathcal{H} is then specified by means of the tuple:

$$(2.2) \quad (\Xi, S, E, G, R),$$

with all the symbols as defined above. Given a *GLSw-system* \mathcal{H} , if $G(e) = \mathbb{R}^{n_i}$ for any $e \in E$, then \mathcal{H} is called *linear switching system* (*LSw-system*) and for simplicity the symbol G is omitted in the tuple (2.2), i.e. $\mathcal{H} = (\Xi, S, E, R)$. A *GLSw-system* \mathcal{H} is said to be *autonomous* if all systems $S(i)$ are autonomous, i.e. $B_i = 0$.

The evolution in time of *GLSw-systems* can be defined as in [15], by means of the notion of *execution*. We recall that a hybrid time basis τ is an infinite or finite sequence of sets $I_j = [t_j, t_{j+1})$, $j = 0, 1, \dots, \text{card}(\tau) - 1$, with $t_{j+1} > t_j$; let be $\text{card}(\tau) = L$. If $L < \infty$, then $t_L = \infty$. Given a hybrid time basis τ , time instants t_j are called *switching times*. Throughout the paper we suppose that given a hybrid time basis, the number of switching times within any bounded time interval is finite, thus avoiding Zeno behaviour [14] in the evolution of the system. Let \mathcal{T} be the set of all hybrid time bases and consider a collection:

$$(2.3) \quad \chi = (\xi_0, \tau, u, \xi, y),$$

where $\xi_0 \in \Xi$ is the initial hybrid state, $\tau \in \mathcal{T}$ is the hybrid time basis, $u \in \mathcal{U}$ is the continuous control input, $\xi : \mathbb{R} \rightarrow \Xi$ is the hybrid state evolution and $y \in \mathcal{Y}$ is the output evolution. The function ξ is defined as follows:

$$\xi(t_0) = \xi_0, \quad \xi(t) = (q(t), x(t)),$$

where at time $t \in I_j$, $q(t) = q(t_j)$, $x(t)$ is the (unique) solution of the dynamical system $S(q(t_j))$, with initial time t_j , initial state $x(t_j)$ and control law u . Moreover, if we set $x^-(t_j) = \lim_{t \rightarrow t_j^-} x(t)$ the following conditions have to be satisfied for any $j = 1, \dots, L - 1$:

$$\begin{aligned} (q(t_{j-1}), q(t_j)) &\in E, \\ x^-(t_j) &\in G(q(t_{j-1}), q(t_j)), \\ x(t_j) &= R(q(t_{j-1}), q(t_j))x^-(t_j). \end{aligned}$$

The output evolution y is defined for any $j = 0, 1, \dots, L - 1$ by:

$$y(t) = C_{q(t_j)}x(t), \quad t \in [t_j, t_{j+1}).$$

A tuple χ of the form (2.3), which satisfies the conditions above, is called an *execution* of \mathcal{H} [14].

2.3. Observability and Detectability. In this section, we introduce the notions of observability and detectability for the class of *GLSw-systems*.

Given a *GLSw-system* \mathcal{H} , we equip the hybrid state space with a metric:

$$\delta((i, x_i), (h, x_h)) = \begin{cases} \infty, & \text{if } i \neq h, \\ \|x_i - x_h\|_{n_i}, & \text{if } i = h. \end{cases}$$

The pair (Ξ, δ) is a metric space.

Definition 2.1. A *GLSw-system* \mathcal{H} is detectable if there exist a control input $\hat{u} \in \mathcal{U}$ and a function $\hat{\xi} : \mathcal{Y} \times \mathcal{U} \rightarrow \Xi$ such that:

$$(2.4) \quad \begin{aligned} &\forall \varepsilon > 0, \forall \rho > 0, \exists \hat{t} > t_0 : \\ &\delta(\hat{\xi}(y|_{[t_0, t]}, \hat{u}|_{[t_0, t]}), \xi(t)) \leq \varepsilon, \\ &\forall t \geq \hat{t}, t \neq t_j, j = 0, 1, \dots, L, \end{aligned}$$

for any execution χ with control input \hat{u} and hybrid initial state $\xi_0 = (i, x_0)$ with $\|x_0\|_{n_i} \leq \rho$. If condition (2.4) holds with $\varepsilon = 0$, then \mathcal{H} is observable.

By Definition 2.1, an observable *GLSw-system* is also detectable. By specializing Definition 2.1 to linear systems, the classical observability and detectability notions are recovered. Note that the reconstruction of the current hybrid state is required at every time $t \geq \hat{t}$ with $t \neq t_j$. Time instants t_j are ruled out as it is for observable linear systems, where the current state may be reconstructed only at every time strictly greater

than the initial time. However, observability and detectability for linear systems are defined independently from the control function, while here we assume to choose a suitable control law. The two definitions coincide for linear systems but not for *GLSw*-systems. In fact, if the observability (or detectability) property were required for *any* input function, then any *GLSw*-system would never be observable (or detectable), see e.g. [8, 1]. However, we will show in Section 3 that if a switching system is observable in the sense of Definition 2.1, then it is observable for “almost all” input functions.

Definition 2.1 requires the reconstruction of the discrete and of the continuous state. We consider these two issues separately, by stating conditions that ensure the reconstruction of the discrete state in Section 3 and of the continuous state in Section 4.

3. LOCATION OBSERVABILITY

In this section, we focus on the reconstruction of the discrete component of the hybrid state *only*. By specializing Definition 2.1, we have:

Definition 3.1. A *GLSw*-system \mathcal{H} is *location observable* if there exist a control input $\hat{u} \in \mathcal{U}$ and a function $\hat{q}: \mathcal{Y} \times \mathcal{U} \rightarrow Q$ such that:

$$(3.1) \quad \begin{aligned} &\forall \rho > 0, \exists \hat{t} > t_0 : \\ &\hat{q}(y|_{[t_0, \hat{t}]}, \hat{u}|_{[t_0, \hat{t}]}) = q(\hat{t}), \\ &\forall t \geq \hat{t}, t \neq t_j, j = 0, 1, \dots, L-1, \end{aligned}$$

for any execution χ with control input \hat{u} and hybrid initial state $\xi_0 = (i, x_0)$ with $\|x_0\|_{n_i} \leq \rho$.

A *GLSw*-system \mathcal{H} is said to be *location observable for a control input* $\hat{u} \in \mathcal{U}$ if there exists a function $\hat{q}: \mathcal{Y} \times \mathcal{U} \rightarrow Q$ such that condition (3.1) is satisfied. The definition of location observability guarantees the reconstruction of the discrete state, but not of the switching times, as the following example shows.

Example 3.2. Consider a *GLSw*-system $\mathcal{H} = (\Xi, S, E, G, R)$, where $\Xi = \{1\} \times \mathbb{R}^3$, $E = \{e\}$ with $e = (1, 1)$ and $G(e) = \mathbb{R}^3$. Let the dynamical system $S(1)$ and the reset function $R(e)$ be described by the following dynamical matrices:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, R(e) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The system \mathcal{H} is trivially location observable for any control input u . However since for any $x \in \mathbb{R}^3$, $(R(e) - I)x$ belongs to the kernel of the observability matrix associated with $S(1)$, it is not possible to reconstruct the switching times, for any choice of the control input u .

For later use, given $i, h \in Q$, define the following augmented linear system S_{ih} :

$$(3.2) \quad \dot{z} = A_{ih}z + B_{ih}u, \quad y_{ih} = C_{ih}z,$$

where:

$$A_{ih} = \begin{pmatrix} A_i & 0 \\ 0 & A_h \end{pmatrix}, \quad B_{ih} = \begin{pmatrix} B_i \\ B_h \end{pmatrix}, \quad C_{ih} = \begin{pmatrix} C_i & -C_h \end{pmatrix}.$$

Let $\mathcal{V}_{ih} \subseteq \mathbb{R}^{n_i+n_h}$ be the maximal controlled invariant subspace [3] for system S_{ih} contained in $\ker(C_{ih})$, i.e. the maximal subspace $F \subseteq \mathbb{R}^{n_i+n_h}$ satisfying the following sets inclusions:

$$(3.3) \quad A_{ih}F \subseteq F + \text{Im}(B_{ih}), \quad F \subseteq \ker(C_{ih}).$$

Define $\hat{J} = \{(i, h) \in Q \times Q : i \neq h\}$ and consider the set:

$$\mathcal{U}^* = \left\{ u \in \mathcal{U} : u \neq \tilde{u}, \text{ a.e., } \forall \tilde{u} \in \tilde{\mathcal{U}} \right\},$$

where:

$$(3.4) \quad \begin{aligned} \tilde{\mathcal{U}} &= \bigcup_{(i,h) \in \hat{J}} \mathcal{U}_{ih}, \\ \mathcal{U}_{ih} &= \left\{ u \in \mathcal{U} : u(t) = K_{ih}z(t) + v_{ih}(t), \right. \\ &\quad \left. t \geq \hat{t}, \text{ for some } \hat{t} \in \mathbb{R} \right\}, \end{aligned}$$

the gain K_{ih} is such that $(A_{ih} + B_{ih}K_{ih})\mathcal{V}_{ih} \subseteq \mathcal{V}_{ih}$, $v_{ih}(t) \in B_{ih}^{-1}(\mathcal{V}_{ih})$, $\forall t \geq \hat{t}$ and $z(t)$ is the state of system S_{ih} at time t , under control u with $z(\hat{t}) \in \mathcal{V}_{ih}$. The set \mathcal{U}^* is composed of the control inputs u such that after a finite time \hat{t} the output y_{ih} of S_{ih} with any initial state $x_0 \in \mathbb{R}^{n_i+n_h}$ and the control input u is not identically zero for any choice of $(i,h) \in \hat{J}$. We will show that control inputs in \mathcal{U}^* ensure the reconstruction of the discrete state. The following result identifies conditions for nonemptiness of \mathcal{U}^* .

Lemma 3.3. *Given a GLSw-system \mathcal{H} , the set \mathcal{U}^* is nonempty if*

$$(3.5) \quad \forall (i,h) \in \hat{J}, \exists k \in \mathbb{N}, k < n_i + n_h : C_i A_i^k B_i \neq C_h A_h^k B_h.$$

The proof of the above result requires some technicalities and is therefore reported in the Appendix. We now have all the ingredients for characterizing location observability of switching systems.

Theorem 3.4. *A GLSw-system \mathcal{H} is location observable if and only if condition (3.5) holds.*

Proof. (Necessity) Suppose by contradiction, that $\exists (i,h) \in \hat{J}$ such that condition (3.5) is not satisfied and consider any $u \in \mathcal{U}$ and any executions $\chi_1 = ((i,0), \tau, u, \xi_1, y_1)$ and $\chi_2 = ((h,0), \tau, u, \xi_2, y_2)$ with $\tau = \{I_0\}$ and $I_0 = [0, \infty)$. It is readily seen that $y_1 = y_2$ and therefore the discrete state cannot be reconstructed. (Sufficiency) By Lemma 3.3, condition (3.5) implies that $\mathcal{U}^* \neq \emptyset$; choose any $u \in \mathcal{U}^*$ and consider any execution $\chi = (\xi_0, \tau, u, \xi, y)$. Consider any $j < L$ and let $\xi(t) = (i, x(t))$, $t \in [t_j, t_{j+1})$. Given any $h \in Q$, denote by $y_{ih}(t, t_j, z, u|_{[t_j, t)})$ the output evolution at time t of system S_{ih} with initial state $z \in \mathbb{R}^{n_i+n_h}$ at initial time t_j and control law $u|_{[t_j, t)}$. Since $u \in \mathcal{U}^*$ then for any $\varepsilon > 0$, for any $h \neq i$ and for any $w \in \mathbb{R}^{n_h}$ there exists a time $t \in (t_j, t_j + \varepsilon)$ such that $y_{ih}(t, t_j, (x(t_j), w)', u) \neq 0$. This implies that $y(t) \neq y_h(t)$, where y_h is the output associated with the execution $(\xi_{0h}, \tau, u, \xi_h, y_h)$ with $\xi_h(t) = (h, x_h(t))$, $t \in [t_j, t_{j+1})$. Hence, the discrete state can be reconstructed for any $t \in (t_j, t_{j+1})$, and the statement follows. \square

It is seen from the above result that *if a GLSw-system \mathcal{H} is location observable then it is location observable for any input function $u \in \mathcal{U}^*$* . A control law that ensures location observability is derived in the proof of Lemma 3.3. Moreover, if the set of control inputs is the set $\mathcal{C}^\infty(\mathbb{R}^m)$ of smooth functions $u : \mathbb{R} \rightarrow \mathbb{R}^m$ (instead of the set \mathcal{U} of piecewise continuous functions), then \mathcal{U}^* contains all and nothing but the control inputs which ensure location observability.

Remark 3.5. Condition (3.5) was first given in [8] as a necessary and sufficient condition for guaranteeing location observability of linear switching systems. A subclass of switching systems was then considered in [1] where similar observability conditions can be found. While the notion of observability of [1] and the one in the present paper (Definition 2.1 or equivalently the definition in [8]) are slightly different, the notions of location observability coincide in the two papers. This translates in a characterization of location observability in [1] which is equivalent to the one in [8] and hence to the one of the present paper (compare Theorem 3 of [1], Theorem 8 of [8] and Theorem 3.4 of this paper).

4. CHARACTERIZING OBSERVABILITY AND DETECTABILITY

Definition 2.1 implies that *a GLSw-system is observable if and only if it is location observable and $S(i)$ is observable for any $i \in Q$* .

The intuitive algorithm for the reconstruction of the (current) hybrid state of an observable GLSw-system \mathcal{H} ,

processes the output $y \in \mathcal{Y}$ and the input $u \in \mathcal{U}^*$. It first reconstructs the current discrete state, by looking for the unique $i \in Q$ such that²

$$(4.1) \quad Y^{(n_i)}(t) \in \text{Im}(\mathcal{O}_i) + \mathcal{F}_i u(t),$$

where $Y^{(n_i)}(t) = (y(t)' \quad \dot{y}(t)' \quad \dots \quad y^{(n_i-1)}(t)')'$, \mathcal{O}_i is the observability matrix associated with $S(i)$ and

$$\mathcal{F}_i = \begin{pmatrix} C_i & 0 & \dots & 0 \\ C_i A_i & C_i B_i & \dots & 0 \\ \dots & \dots & \dots & 0 \\ C_i A_i^{n_i} & C_i A_i^{n_i-1} B & \dots & C_i B_i \end{pmatrix};$$

Then, on the basis of the knowledge of i , it reconstructs the current continuous state $x(t)$, by computing:

$$(4.2) \quad \{x(t)\} = \mathcal{O}_i^{-1} \left(Y^{(n_i)}(t) - \mathcal{F}_i u(t) \right).$$

We now focus on *LSw*-systems and derive conditions that ensure detectability. Since location observability is a necessary condition for a switching system to be observable or detectable, we assume now that this property holds for all systems considered in this section. Given a *LSw*-system $\mathcal{H} = (\Xi, S, E, R)$, define the autonomous *LSw*-system:

$$(4.3) \quad \mathcal{H}' = (\Xi, S', E, R),$$

where $S'(i)$ is defined as $S(i)$ in (2.1) with $B_i = 0$. We assume that \mathcal{H}' is with full discrete evolution information, i.e. that the discrete state and the switching times are known at any time. Clearly, detectability of \mathcal{H} implies detectability of \mathcal{H}' . Under some appropriate conditions, the converse implication is true:

Lemma 4.1. *A location observable LSw-system \mathcal{H} is detectable if \mathcal{H}' is detectable and \mathcal{H} satisfies the following property:*

$$(4.4) \quad E^\circ = \emptyset \quad \text{or} \quad \text{Im}(R(e) - I) \cap \ker(\mathcal{O}_i) = \{0\}, \forall e \in E^\circ,$$

where $E^\circ = \{(i, h) \in E : i = h\}$ and \mathcal{O}_i is the observability matrix associated with $S(i)$.

Under condition (4.4), if a transition $(i, i) \in E^\circ$ occurs in \mathcal{H} at time t_j from a hybrid state (i, x^-) to a hybrid state (i, x^+) with $x^+ = R(i, i)x^- \neq x^-$ then $x^+ - x^- \notin \ker(\mathcal{O}_i)$. Hence the switching time t_j can be reconstructed³. Then, the proof of the result above just follows from the linearity of the continuous dynamics in \mathcal{H} and from the definition of \mathcal{H}' .

The result of Lemma 4.1 reduces the analysis of detectability of a linear switching system *with control*, to that of an *autonomous* linear switching system.

For analyzing detectability of \mathcal{H}' it is useful to first perform a discrete state space decomposition.

Given $\mathcal{H}' = (\Xi, S', E, R)$ as in (4.3) and a set $\hat{Q} \subseteq Q$ let

$$\mathcal{H}'|_{\hat{Q}} = (\Xi|_{\hat{Q}}, S'|_{\hat{Q}}, E|_{\hat{Q}}, R|_{\hat{Q}}),$$

be the switching sub-system of \mathcal{H}' obtained by restricting the discrete state space Q of \mathcal{H} to \hat{Q} , i.e. such that $\Xi|_{\hat{Q}} = \bigcup_{i \in \hat{Q}} \{i\} \times \mathbb{R}^{n_i}$, $S'|_{\hat{Q}}(i) = S'(i)$, $E|_{\hat{Q}} = \{(i, h) \in E : i, h \in \hat{Q}\}$ and $R|_{\hat{Q}}(i, h) = R(i, h)$.

Proposition 4.2. *The LSw-system \mathcal{H}' is detectable if and only if the LSw-system $\mathcal{H}'|_{\hat{Q}}$ with $\hat{Q} = \{i \in Q : S(i) \text{ is not observable}\}$ is detectable.*

Proof. (Necessity) Obvious. (Sufficiency) Consider any execution χ of \mathcal{H}' . If $q(t) \in \hat{Q}$ for any time $t \geq t_0$ then the detectability of $\mathcal{H}'|_{\hat{Q}}$ implies the asymptotic reconstruction of the hybrid state evolution of χ . If $q(t) \notin \hat{Q}$ for some finite time t , then $S'(q(t))$ is observable and hence it is possible to (exactly) reconstruct the

²If the switching system \mathcal{H} is location observable and $u \in \mathcal{U}^*$, Theorem 3.4 guarantees that such discrete state i is unique.

³Note that the switching system of Example 3.2 does not satisfy condition (4.4) and therefore switching times in that case cannot be reconstructed.

continuous state of \mathcal{H}' in infinitesimal time. Once the continuous state $x(t')$ is known at time $t' > t$, location observability of \mathcal{H}' ensures the reconstruction of the hybrid state for any time $t'' \geq t'$ with $t'' \neq t_j$. \square

By Proposition 4.2 there is no loss of generality in assuming that system $S'(i)$ is not observable for any $i \in Q$. Moreover, we assume that $S'(i)$, $i \in Q$, are in observability canonical form, i.e. that dynamical matrices associated with $S'(i)$ are of the form:

$$A_i = \begin{pmatrix} A_i^{(11)} & 0 \\ A_i^{(21)} & A_i^{(22)} \end{pmatrix}, C_i = \begin{pmatrix} C_i^{(1)} & 0 \end{pmatrix},$$

where $A_i^{(22)} \in \mathbb{R}^{d_i \times d_i}$, $0 < d_i \leq n_i$ matrices $A_i^{(11)}$, $A_i^{(21)}$ are of appropriate dimensions and $(A_i^{(11)}, C_i^{(1)})$ is an observable matrix pair, for any $i \in Q$. This assumption is made without loss of generality: suppose that, for some $i \in Q$, the dynamical matrices A_i, C_i of the switching system \mathcal{H}' are not in the observability canonical form. Then, we define an invertible linear transformation $T_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_i}$ such that $T_i A_i T_i^{-1}$ and $C_i T_i^{-1}$ are in the observability canonical form. For all $j \in Q$ such that the dynamical matrices A_j, C_j of the switching system \mathcal{H}' are in the observability canonical form, we let T_j be the identity matrix. We then define the *hybrid state space transformation* $T : \Xi \rightarrow \Xi$ such that for any $(i, x) \in \Xi$, $T(i, x) := (i, T_i x)$. The reset function in the new coordinates is given by $T_h R(e) T_i^{-1}$, for any $e = (i, h) \in E$. The continuous component x of the hybrid state (i, x) of \mathcal{H}' can be partitioned as $x = \begin{pmatrix} x_1' & x_2' \end{pmatrix}'$, with $x_1 \in \mathbb{R}^{n_i - d_i}$, $x_2 \in \mathbb{R}^{d_i}$, and the reset matrix $R(e)$ can be partitioned as:

$$R(e) = \begin{pmatrix} R^{(11)}(e) & R^{(12)}(e) \\ R^{(21)}(e) & R^{(22)}(e) \end{pmatrix},$$

where $R^{(22)}(e) \in \mathbb{R}^{d_h \times d_i}$ and $R^{(11)}(e)$, $R^{(12)}(e)$, $R^{(21)}(e)$ are of appropriate dimensions. Given the *LSw*-system \mathcal{H}' as in (4.3), define the *GLSw*-system:

$$(4.5) \quad \mathcal{H}_0 = (\Xi_0, S_0, E, G_0, R_0),$$

where:

- $\Xi_0 = \bigcup_{i \in Q} \{i\} \times \mathbb{R}^{d_i}$;
- $S_0(i)$ is described by dynamics $\dot{z}(t) = A_i^{(22)} z(t)$, for any $i \in Q$;
- $G_0(e) = \ker(R^{(12)}(e))$, for any $e \in E$;
- $R_0(e) = R^{(22)}(e)$, for any $e \in E$.

There is a strong connection between detectability of \mathcal{H}' and asymptotic stability of \mathcal{H}_0 . Set $\mathcal{B} := \bigcup_{i \in Q} \{i\} \times \mathcal{B}_i$, where $\mathcal{B}_i = \{x \in \mathbb{R}^{n_i} : \|x\|_{n_i} \leq 1\}$. We also define $\varepsilon \mathcal{B} := \bigcup_{i \in Q} \{i\} \times \varepsilon \mathcal{B}_i$ for any $\varepsilon \in \mathbb{R}^+$. An autonomous *GLSw*-system \mathcal{H} is *asymptotically stable* if the continuous component of the hybrid state of any execution χ of \mathcal{H} converges to the origin as time goes to infinity, or equivalently:

$$\forall \varepsilon > 0, \forall \rho > 0, \exists \hat{t} > t_0 : \xi(t) \in \varepsilon \mathcal{B}, \forall t \geq \hat{t},$$

for any execution χ with hybrid initial state $\xi_0 \in \rho \mathcal{B}$. The following holds:

Proposition 4.3. *The LSw-system \mathcal{H}' is detectable if and only if the GLSw-system \mathcal{H}_0 is asymptotically stable.*

Proof. (Sketch.) Let \mathcal{E}_0 be the set of executions of \mathcal{H}' such that $C_{q(t)} x(t) = 0, \forall t \geq t_0$. The continuous component $x(t)$ of the hybrid state $(q(t), x(t))$ of any execution in \mathcal{E}_0 belongs to the subspace $\ker(\mathcal{O}_i)$ with $i = q(t)$ for any $t \in I_j$ and $j = 0, 1, \dots, L$. By definition of \mathcal{E}_0 , \mathcal{H}' is detectable if and only if the continuous component of the hybrid state ξ of any $\chi \in \mathcal{E}_0$ converges to the origin, i.e. $\forall \varepsilon > 0, \forall \rho > 0, \exists \hat{t} \geq t_0$ such that $\xi(t) \in \varepsilon \mathcal{B}, \forall t \geq \hat{t}$, for any $\chi \in \mathcal{E}_0$ with hybrid initial state $\xi_0 \in \rho \mathcal{B}$. By definition of the observability canonical form, this is equivalent to asymptotic stability of \mathcal{H}_0 . \square

By combining Lemma 4.1 and Propositions 4.2 and 4.3 we obtain the following characterization of detectability of LSw -systems.

Theorem 4.4. *A LSw -system \mathcal{H} is detectable if the following conditions are satisfied:*

- i): \mathcal{H} is location observable, and
- ii): \mathcal{H} satisfies condition (4.4), and
- iii): \mathcal{H}_0 is asymptotically stable.

Conversely, if \mathcal{H} is detectable then conditions i) and iii) are satisfied.

Since the executions associated with a $GLSw$ -system (Ξ, S, E, G, R) are also executions of the LSw -system $\mathcal{H} = (\Xi, S, E, R)$, the conditions of Theorem 4.4 are also sufficient for a $GLSw$ -system to be detectable.

Detectability of switching systems has also been addressed in [8]. The above result provides a deeper analysis than the one in [8] since it reduces detectability of LSw -systems to asymptotic stability of $GLSw$ -systems (compare Theorem 9 of [8] with the above result). This allows one to leverage the rich literature on stability of hybrid systems (see e.g. [17, 5, 12] and the references therein) for checking detectability. While checking conditions i) and ii) is straightforward, checking condition iii) requires the analysis of asymptotic stability of switching systems with guards.

We now derive sufficient conditions for assessing the asymptotic stability of \mathcal{H}_0 , by *abstracting* \mathcal{H}_0 with linear switching systems with no guards. Given the autonomous $GLSw$ -system \mathcal{H}_0 as in (4.5) define the following autonomous LSw -systems:

$$(4.6) \quad \mathcal{H}_1 = (\Xi_0, S_0, E, R_0), \quad \mathcal{H}_2 = (\Xi_0, S_0, E, R_2),$$

where $R_2(e) = R^{(22)}(e)\pi_{\ker(R^{(12)}(e))}$. The following result holds:

Proposition 4.5. *The autonomous $GLSw$ -system \mathcal{H}_0 is asymptotically stable if either \mathcal{H}_1 or \mathcal{H}_2 is asymptotically stable.*

Since transitions in LSw -systems \mathcal{H}_1 and \mathcal{H}_2 are independent of the continuous state, the asymptotic stability analysis of \mathcal{H}_1 and \mathcal{H}_2 is in general easier than the one of \mathcal{H}_0 . An application of this result is shown in the next section.

5. AN ILLUSTRATIVE EXAMPLE

In this section, we present an example that shows the interest and applicability of our results. Consider the linear switching system $\mathcal{H} = (\Xi, S, E, R)$, where:

- $\Xi = (\{1\} \times \mathbb{R}^4) \cup (\{2\} \times \mathbb{R}^3) \cup (\{3\} \times \mathbb{R}^2) \cup (\{4\} \times \mathbb{R}) \cup (\{5\} \times \mathbb{R}^3) \cup (\{6\} \times \mathbb{R}^2)$;
- S associates to any $i \in Q = \{1, 2, 3, 4, 5, 6\}$ the linear control system $S(i)$ of (2.1), where:

$$\begin{aligned}
A_1 &= \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}, & B_1 &= \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, & C_1 &= (1 \ 1 \ 0 \ 0), \\
A_2 &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix}, & B_2 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, & C_2 &= (1 \ 0 \ 0), \\
A_3 &= \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, & B_3 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & C_3 &= (1 \ 0), \\
A_4 &= 3, & B_4 &= 1, & C_4 &= 1, \\
A_5 &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -2 \end{pmatrix}, & B_5 &= \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, & C_5 &= (1 \ 0 \ 0), \\
A_6 &= \begin{pmatrix} 5 & 0 \\ 2 & -3 \end{pmatrix}, & B_6 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & C_6 &= (1 \ 0);
\end{aligned}$$

- $E = \{(1, 2), (2, 1), (2, 3), (2, 5), (3, 3), (3, 6), (4, 1), (4, 2), (5, 4), (5, 6), (6, 5)\}$;
- R is defined by:

$$\begin{aligned}
R(1, 2) &= \begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & R(2, 1) &= \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
R(2, 3) &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, & R(2, 5) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
R(3, 3) &= \begin{pmatrix} 1 & 1 \\ 0 & 10 \end{pmatrix}, & R(3, 6) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
R(4, 1) &= (1 \ -1 \ 0 \ 2)', & R(4, 2) &= (1 \ 1 \ 1)', \\
R(5, 4) &= (1 \ 1 \ 1), & R(5, 6) &= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 10 & 0 \end{pmatrix}, \\
R(6, 5) &= \begin{pmatrix} 1 & 0 \\ 0 & 10 \\ 0 & 10 \end{pmatrix}.
\end{aligned}$$

The Finite State Machine associated with system \mathcal{H} is depicted in Figure 1. Let us analyze observability and

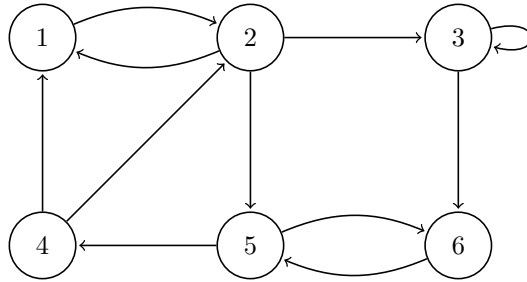


FIGURE 1. Finite State Machine associated with the linear switching system \mathcal{H} .

detectability properties of the linear switching system \mathcal{H} . The linear systems $S(i)$ associated with discrete states $i = 1, 2, 3, 5, 6$ are detectable but not observable and therefore we conclude that \mathcal{H} is not observable.

We now check detectability of \mathcal{H} . For this purpose, we apply Theorem 4.4. We start by checking condition i). The Markov parameters associated with systems $S(i)$, $i \in Q$ are given for any $k \in \mathbb{N}$ by:

$$(5.1) \quad \begin{aligned} C_1 A_1^k B_1 &= 1, & C_2 A_2^k B_2 &= 2^k, & C_3 A_3^k B_3 &= 0, \\ C_4 A_4^k B_4 &= 3^k, & C_5 A_5^k B_5 &= 4, & C_6 A_6^k B_6 &= 5^k. \end{aligned}$$

Hence, condition (3.5) is satisfied for $k = 1$. Thus by Theorem 3.4, the linear switching system \mathcal{H} is location observable. We now check condition ii) of Theorem 4.4. In this case $E^\circ = \{(3, 3)\}$ and $Im(R(3, 3) - I) \cap \ker(\mathcal{O}_3) = \{0\}$; thus condition ii) is satisfied. Finally, we check condition iii). Since the linear system $S(4)$ is observable, by Proposition 4.2 the switching system \mathcal{H}' associated with \mathcal{H} is detectable if and only if $\mathcal{H}'|_{\hat{Q}}$ with $\hat{Q} = \{1, 2, 3, 5, 6\}$, is detectable. The resulting linear switching system $\mathcal{H}'|_{\hat{Q}}$ is characterized by the Finite State Machine in Figure 2.

We can now introduce the *GLSw*-system \mathcal{H}_0 of (4.5) associated with $\mathcal{H}'|_{\hat{Q}}$:

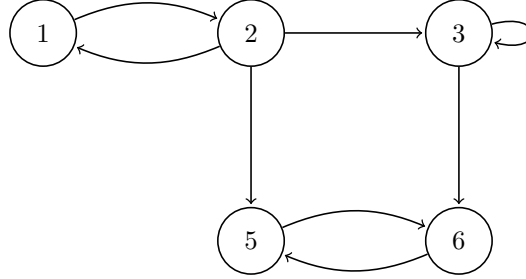


FIGURE 2. Finite State Machine associated with the linear switching system $\mathcal{H}'|_{\hat{Q}}$, with $\hat{Q} = \{1, 2, 3, 5, 6\}$.

$$(5.2) \quad \mathcal{H}_0 = (\Xi_0, S_0, E, G_0, R_0),$$

where:

- $\Xi_0 = (\{1\} \times \mathbb{R}^2) \cup (\{2\} \times \mathbb{R}^2) \cup (\{3\} \times \mathbb{R}) \cup (\{5\} \times \mathbb{R}^2) \cup (\{6\} \times \mathbb{R})$;
- $S_0(i)$ is described for any $i \in \hat{Q}$ by dynamics $\dot{z}(t) = A_i^{(22)} z(t)$, where:

$$\begin{aligned} A_1^{(22)} &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, A_2^{(22)} = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, & A_3^{(22)} &= -1, \\ A_5^{(22)} &= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, A_6^{(22)} &= -3 \end{aligned}$$

- $E = \{(1, 2), (2, 1), (2, 3), (2, 5), (3, 3), (3, 6), (5, 6), (6, 5)\}$;
- $G_0(i, h) = \ker(R^{(12)}(i, h))$ for any $(i, h) \in E$, where:

$$\begin{aligned} R^{(12)}(1, 2) &= \begin{pmatrix} 2 & -3 \end{pmatrix}, & R^{(12)}(2, 1) &= \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \\ R^{(12)}(2, 3) &= \begin{pmatrix} -1 & 0 \end{pmatrix}, & R^{(12)}(2, 5) &= \begin{pmatrix} 0 & 0 \end{pmatrix}, \\ R^{(12)}(3, 3) &= 1, & R^{(12)}(3, 6) &= 0, \\ R^{(12)}(5, 6) &= \begin{pmatrix} 1 & 0 \end{pmatrix}, & R^{(12)}(6, 5) &= 0; \end{aligned}$$

- $R_0(i, h) = R^{(22)}(i, h)$ for any $(i, h) \in E$, where:

$$\begin{aligned} R^{(22)}(1, 2) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & R^{(22)}(2, 1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ R^{(22)}(2, 3) &= \begin{pmatrix} 1 & 1 \end{pmatrix}, & R^{(22)}(2, 5) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ R^{(22)}(3, 3) &= 10, & R^{(22)}(3, 6) &= 1, \\ R^{(22)}(5, 6) &= \begin{pmatrix} 10 & 0 \end{pmatrix}, & R^{(22)}(6, 5) &= \begin{pmatrix} 10 \\ 10 \end{pmatrix}. \end{aligned}$$

The Finite State Machine associated with \mathcal{H}_0 (Figure 2) is composed by three strongly connected components⁴, i.e. one involving the discrete states $1, 2 \in \hat{Q}$, the one involving the discrete state $3 \in \hat{Q}$ and the other involving the discrete states $5, 6 \in \hat{Q}$. It is well-known that the asymptotic stability of a switching system can be assessed by studying this property in each strongly connected component. More precisely, \mathcal{H}_0 is asymptotically stable if and only if switching system $\mathcal{H}_0|_{Q_1}$, with $Q_1 = \{1, 2\}$, switching system $\mathcal{H}_0|_{Q_2}$, with $Q_2 = \{3\}$, and switching system $\mathcal{H}_0|_{Q_3}$, with $Q_3 = \{5, 6\}$ are asymptotically stable.

We first consider $\mathcal{H}_0|_{Q_1}$. We recall from [12] that an autonomous *GLSw*-system (with identity reset map) is asymptotically stable if it admits a common Lyapunov function V . By defining for any $x \in \mathbb{R}^2$ the function $V(x) = x'Px$ with $P = I$, we obtain:

$$(A_1^{(22)})'P + PA_1^{(22)} \leq -Q, \quad (A_2^{(22)})'P + PA_2^{(22)} \leq -Q,$$

where:

$$Q = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix} \geq 0.$$

Hence V is a common Lyapunov function for sub-systems $S_0(1)$ and $S_0(2)$ of $\mathcal{H}_0|_{Q_1}$ and by Theorem 2.1 in [12] we conclude that $\mathcal{H}_0|_{Q_1}$ is asymptotically stable⁵.

Let us consider $\mathcal{H}_0|_{Q_2}$. The *GLSw*-system $\mathcal{H}_0|_{Q_2}$ is characterized by dynamical matrix $A_3^{(22)} = -1$, reset matrix $R_2(3, 3) = 0$ and guard $G_0(3, 3) = \{0\}$ and hence it is asymptotically stable.

Let us now consider $\mathcal{H}_0|_{Q_3}$ and let us apply Proposition 4.5 to investigate stability properties of $\mathcal{H}_0|_{Q_3}$. It is readily seen that the abstraction \mathcal{H}_1 of (4.6), that corresponds to $\mathcal{H}_0|_{Q_3}$ is unstable. Let us now consider the abstraction \mathcal{H}_2 of $\mathcal{H}_0|_{Q_3}$. The reset map $R_2(e)$ with $e = (5, 6)$ associated to \mathcal{H}_2 is given by:

$$R_2(5, 6) = R^{(22)}(5, 6)\pi_{\ker(R^{(12)}(5, 6))} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Therefore, since dynamical matrices $A_5^{(22)}$ and $A_6^{(22)}$ are Hurwitz it is easy to see that the *LSw*-system \mathcal{H}_2 is asymptotically stable. Thus, by Proposition 4.5 also $\mathcal{H}_0|_{Q_3}$ is asymptotically stable.

We conclude that the switching system \mathcal{H}_0 is asymptotically stable and therefore condition iii) of Theorem 4.4 is satisfied. Hence, by Theorem 4.4, the linear switching system \mathcal{H} is detectable.

6. CONCLUSIONS

We addressed observability and detectability of linear switching systems. We derived a computable necessary and sufficient condition for a switching system to be observable. Further, we derived a Kalman decomposition of the switching system, which reduces detectability of linear switching systems to asymptotic stability of suitable linear switching systems with guards associated with the original systems. The study of detectability is a fundamental step towards the design of a hybrid observer. In fact, by Definition 2.1, a necessary condition for the existence of a hybrid observer for a *LSw*-system \mathcal{H} is that \mathcal{H} is detectable. On the other hand, as shown in Section 3, observability of \mathcal{H} implies the existence of an algorithm that reconstructs the current hybrid state; in particular, the combination of (4.1) and (4.2) can be thought of as a hybrid observer. However,

⁴We recall that a strongly connected component of a FSM is a FSM, with a path between any two discrete states.

⁵Dynamical matrices $A_1^{(22)}$ and $A_2^{(22)}$ have been taken from [17].

such an observer requires an infinite precision in the computation of the vector $Y^{(n)}(t)$. Further work will identify appropriate conditions on linear switching systems, for the existence and design of hybrid observers.

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7. APPENDIX: PROOF OF LEMMA 3.3

We first need two preliminary technical lemmas.

Lemma 7.1. *If condition (3.5) is satisfied then for any $(i, h) \in \hat{J}$, $B_{ih}^{-1}(\mathcal{V}_{ih}) \neq \mathbb{R}^m$.*

Proof. By contradiction, suppose that $B_{ih}^{-1}(\mathcal{V}_{ih}) = \mathbb{R}^m$ for some $(i, h) \in \hat{J}$. Then $\text{Im}(B_{ih}) \subseteq \mathcal{V}_{ih}$ and by (3.3), $A_{ih}\mathcal{V}_{ih} \subseteq \mathcal{V}_{ih} + \text{Im}(B_{ih}) \subseteq \mathcal{V}_{ih}$, i.e. \mathcal{V}_{ih} is A_{ih} -invariant and contains $\text{Im}(B_{ih})$. Since the minimal A_{ih} -invariant subspace containing $\text{Im}(B_{ih})$ is $\text{Im}(B_{ih} \ A_{ih}B_{ih} \ \dots \ A_{ih}^{n-1}B_{ih})$, with $n = n_i + n_h$, then $\text{Im}(B_{ih} \ A_{ih}B_{ih} \ \dots \ A_{ih}^{n-1}B_{ih}) \subseteq \mathcal{V}_{ih} \subseteq \ker(C_{ih})$, which implies $C_{ih}(B_{ih} \ A_{ih}B_{ih} \ \dots \ A_{ih}^{n-1}B_{ih}) = 0$. Thus condition (3.5) is not satisfied and hence a contradiction holds. \square

Lemma 7.2. *Let $\{M_i \in \mathbb{R}^{m \times n^T}, i \in Q\}$ be a family of nonzero matrices. There exists $z \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that $M_i z \neq 0, \forall i \in Q$, where*

$$(7.1) \quad \mathbf{z}' = (z' \ \lambda z' \ \lambda^2 z' \ \dots \ \lambda^{T-1} z')'.$$

Proof. By setting $M_i = (M_{i0} \ M_{i1} \ \dots \ M_{iT-1})$ and $M_i(\lambda) = M_{i0} + \lambda M_{i1} + \lambda^2 M_{i2} + \dots + \lambda^{T-1} M_{iT-1}$, with $M_{ij} \in \mathbb{R}^{m \times n}$, for any $z \in \mathbb{R}^n$, $M_i z = M_i(\lambda)z$. Given $i \in Q$, since $M_i \neq 0$, there are a finite number of values θ such that $M_i(\theta) = 0$. Choose λ such that $M_i(\lambda) \neq 0, \forall i \in Q$. Then there exists $z \notin \bigcup_{i \in Q} \ker(M_i(\lambda))$ which implies $M_i z \neq 0, \forall i \in Q$. \square

We now give the proof of Lemma 3.3.

Proof. By contradiction, suppose that the set \mathcal{U}^* is empty and let be $n = n_i + n_h$. Then

$$(7.2) \quad \begin{aligned} &\forall u \in \mathcal{U}, \exists t', t'' \in \mathbb{R}, \exists (i, h) \in \hat{J} \text{ and } \tilde{u} \in \mathcal{U}_{ih} \text{ s.t.} \\ &u(t) = \tilde{u}(t), \forall t \in [t', t'']. \end{aligned}$$

Let V^{ih} be the set of smooth functions $v : \mathbb{R} \rightarrow B_{ih}^{-1}(\mathcal{V}_{ih})$ and let $\hat{\mathcal{U}} \subset \mathcal{U}$ be the set of smooth, not identically zero functions. By definition of \mathcal{U}_{ih} , condition (7.2) implies:

$$(7.3) \quad \begin{aligned} &\forall u \in \hat{\mathcal{U}}, \exists t', t'' \in \mathbb{R}, \exists (i, h) \in \hat{J}, \bar{z} \in \mathcal{V}_{ih} \text{ and } v_{ih} \in V^{ih} \\ &\text{s.t. } u(t) = K_{ih}z(t) + v_{ih}(t), \forall t \in [t', t''], \end{aligned}$$

where $\dot{z}(t) = \hat{A}_{ih}z(t) + B_{ih}v_{ih}(t)$, $\hat{A}_{ih} = A_{ih} + B_{ih}K_{ih}$ and $z(t') = \bar{z} \in \mathcal{V}_{ih}$. Condition (7.3) implies:

$$(7.4) \quad \forall u \in \hat{\mathcal{U}}, \exists t' \in \mathbb{R}, \exists (i, h) \in \hat{J} \text{ s.t. } \forall \bar{N} \geq 0$$

$$\begin{pmatrix} u(t') \\ \dot{u}(t') \\ \vdots \\ u^{(\bar{N})}(t') \end{pmatrix} \in \mathbf{M}_{ih}^{\bar{N}} \mathcal{V}_{ih} + \mathbf{F}_{ih}^{\bar{N}} (F_{ih} \times F_{ih} \times \dots \times F_{ih}),$$

where $F_{ih} = B_{ih}^{-1}(\mathcal{V}_{ih})$ and

$$\begin{aligned} \mathbf{M}_{ih}^{\bar{N}} &= \begin{pmatrix} K_{ih} \\ K_{ih}\hat{A}_{ih} \\ \vdots \\ K_{ih}\hat{A}_{ih}^{\bar{N}} \end{pmatrix} \in \mathbb{R}^{m(\bar{N}+1) \times \bar{N}}, \\ \mathbf{F}_{ih}^{\bar{N}} &= \begin{pmatrix} I & 0 & \dots & 0 \\ K_{ih}B_{ih} & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K_{ih}\hat{A}_{ih}^{\bar{N}-1}B_{ih} & K_{ih}\hat{A}_{ih}^{\bar{N}-2}B_{ih} & \dots & I \end{pmatrix} \in \mathbb{R}^{m\bar{N} \times m\bar{N}}. \end{aligned}$$

The matrix $\mathbf{F}_{ih}^{\bar{N}}$ is nonsingular. By setting $\dim(F_{ih}) = \nu$, one obtains:

$$\dim(\mathbf{F}_{ih}^{\bar{N}} (F_{ih} \times F_{ih} \times \dots \times F_{ih})) = \nu(\bar{N} + 1),$$

and since (3.5) holds, $\dim(\mathbf{M}_{ih}^{\bar{N}} \mathcal{V}_{ih}) < n$; thus

$$\dim(\mathbf{M}_{ih}^{\bar{N}} \mathcal{V}_{ih} + \mathbf{F}_{ih}^{\bar{N}} (F_{ih} \times F_{ih} \times \dots \times F_{ih})) \leq \nu(\bar{N} + 1) + n.$$

Therefore since by Lemma 7.1, $\nu < m$, we obtain that $\nu(\bar{N} + 1) + n < m(\bar{N} + 1)$ for any $\bar{N} > \frac{n}{m-\nu} - 1$; thus $\mathbf{M}_{ih}^{\bar{N}} \mathcal{V}_{ih} + \mathbf{F}_{ih}^{\bar{N}}(F_{ih} \times F_{ih} \times \dots \times F_{ih})$ is a proper subspace of $\mathbb{R}^{m(\bar{N}+1)}$. Hence there exists a sufficiently large \bar{N} such that the set $\mathbf{M}_{ih}^{\bar{N}} \mathcal{V}_{ih} + \mathbf{F}_{ih}^{\bar{N}}(F_{ih} \times F_{ih} \times \dots \times F_{ih})$ is a proper subspace of $\mathbb{R}^{m(\bar{N}+1)}$ for any $(i, h) \in \hat{J}$. Given some $z \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$ let be $\mathbf{u}(t) = z \exp(\lambda t) \in \hat{\mathcal{U}}$. It follows that:

$$\begin{pmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \\ \vdots \\ \mathbf{u}^{(\bar{N})}(t) \end{pmatrix} = \begin{pmatrix} z \\ \lambda z \\ \vdots \\ \lambda^{T-1} z \end{pmatrix} \exp(\lambda t).$$

Set $\mathbf{M}_{ih}^{\bar{N}} \mathcal{V}_{ih} + \mathbf{F}_{ih}^{\bar{N}}(F_{ih} \times F_{ih} \times \dots \times F_{ih}) = \ker(G_{ih})$, for some matrix G_{ih} . By Lemma 7.2 there exist z and λ such that $G_{ih} \mathbf{z} \neq 0$, $\forall (i, h) \in \hat{J}$ where \mathbf{z} is as in (7.1). This implies that the vector

$$\begin{pmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \\ \vdots \\ \mathbf{u}^{(\bar{N})}(t) \end{pmatrix} = \mathbf{z} \exp(\lambda t)$$

does not belong to $\mathbf{M}_{ih}^{\bar{N}} \mathcal{V}_{ih} + \mathbf{F}_{ih}^{\bar{N}}(F_{ih} \times F_{ih} \times \dots \times F_{ih})$, for all $(i, h) \in \hat{J}$ and $t \in \mathbb{R}$, and hence condition (7.4) is false; thus the result follows. \square

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